Structure vs. Randomness

Manin's Conjecture & Szemerédi Regularity

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Overview

Question

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Difficult to answer in general, but one common approach is to break the object into a "structured" component and a "pseudo-random" component.

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Because of this **negligibility**, we would like to be able to easily locate the structured and pseudorandom components of a given object.

> **Typical conjecture:** "Natural" objects behave pseudorandomly after accounting for all the obvious structures.

These conjectures can be extremely hard to prove!

- The primes should behave randomly after accounting for "local" (mod p) obstructions. (Hardy-Littlewood prime tuples conjecture; Riemann hypothesis; ...)
- Solutions to highly nonlinear systems should behave randomly after accounting for conservation laws etc. (Rigorous statistical mechanics; ?Navier-Stokes global regularity?; ...)
- There should exist "describable" algorithms which behave "unpredictably". ($P = BPP$; ?$P \neq NP$?; ...)

Manin's Conjecture

Setup:

- X is a suitable Fano variety over a number field k.
- $\mathcal{L}$ is an ample line bundle on X, and $H_{\mathcal{L}}\colon X(k) \to \mathbb{R}_{\geq 0}$ is the height function associated to $\mathcal{L}$.
- For any subset $Q \subset X(k)$, define the counting function

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- For any subset $Q \subset X(k)$, define the counting function

$$N_{\mathcal{L}}(Q, B) := \#\{x \in Q \mid H_{\mathcal{L}}(x) \leq B\}.$$

Then, there exists an exceptional set $Z \subset X(k)$ such that

$$N_{\mathcal{L}}(X(k) \setminus Z, B) \sim cB^{a} \log B^{b-1}, \qquad B \to \infty,$$

where a and b are geometric invariants of X, and c is Peyre's constant.

Key Image

Manin's Conjecture

- Peyre's Equidistribution: Rational points of Fano varieties should be uniformly distributed throughout the variety except for some local region of chaos ("exceptional thin set") which witnesses dense clusters of rational points that throw off the prediction by the asymptotic formula.
- Subtle relationship between the combinatorics (# of k-rational points of X, with respect to height) and geometric invariants associated to X.

Key Image

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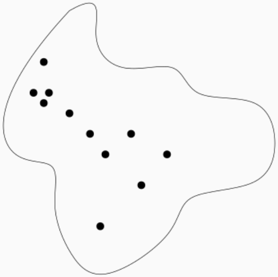
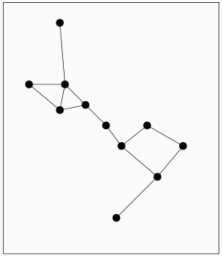


Rational Points on a Variety



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Dense clusters of edges in the graph correspond to dense clusters of rational points in the original variety. But now we now have a new language for asking questions about the distribution of rational points – e.g. presence/absence of certain graph configurations, pseudo-random behaviour of edges etc.

Interesting to ask how we might reflect these graph-theoretic questions back into the geometry of the variety, and if this will point out some new insights. This study group will investigate the potential interplay between the two perspectives, guided by the fundamental dichotomy between structure vs. randomness.

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This connection between Manin's Conjecture & graph theory appears to be new. As the study group progresses, our understanding of the underlying mechanics will also improve, and natural test problems should emerge in the process.

Arithmetic Geometry & Rational Points

Basic Questions about Rational Points

Let $X/\mathbb{Q}$ be a projective variety (= a system of homogeneous equations with $\mathbb{Q}$-coefficients). Rational points on X are (equivalence classes of) rational solutions to these equations.[1] We denote $X(\mathbb{Q})$ to be the set of $\mathbb{Q}$-rational points.

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Fundamental Themes

1) **Existence:** When is $X(\mathbb{Q}) \neq \emptyset$?
2) **Density:** Is $X(\mathbb{Q})$ Zariski dense in X? (or potentially dense, or dense in analytic topologies etc.)
3) **Heights:** If $X(\mathbb{Q})$ is dense, what is the asymptotic distribution of rational points (of bounded height) on the variety?

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Classification of Smooth Algebraic Varieties

Classification with respect to the canonical divisor K_X:

Fano: $-K_X$ is ample.

General Type: K_X is ample;

Intermediate Type: None of the above.

$\dim$	**Fano**	**Intermediate Type**	**General Type**
1	$\mathbb{P}^1$	Elliptic Curves	C, $g(C) \geq 2$
2	$\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, blowups of $\mathbb{P}^2$	$K3$ surfaces, $X_4 \subset \mathbb{P}^3$, Abelian Surfaces, ...	...
3	Smooth: $\sim$ 120 families Singular: $\overline{M}_{0,6}$, ...	Calabi-Yau Varieties	...

Canonical Line Bundle

Let X be a smooth projective variety of dimension n over field k.

The **canonical sheaf**, denoted ω_X, is the highest wedge of the sheaf of differential forms

$$\omega_X = \bigwedge^n \Omega_{X/k}.$$

A **canonical divisor**[a] K_X is any Cartier divisor satisfying the property

$$\mathcal{O}_X(K_X) = \omega_X.$$

More concretely[b], we can think of it as

$$K_X = \sum_D ord_D(\omega_X) \cdot D.$$

[a] One typically refers to "the" canonical divisor even though K_X is only unique up to linear equivalence.

[b] For the case of curves, see [Sil09, §II.4] for an exposition.

Ex: Projective Space

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Ex: Hypersurfaces

Let $X_d \subset \mathbb{P}^n$ be a hypersurface of degree d. Then $\omega_{X_d} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1+d)|_X$.
[Why? Take the top exterior powers of conormal bundle sequence

$$0 \to \Omega_{\mathbb{P}^n}(-d)|_X \to \Omega_{\mathbb{P}^n}|X \to \Omega_X \to 0. \quad]$$

Alternatively, $-K_X = (n+1-d) \cdot H$.

Ample Line Bundle

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- $\mathcal{L}$ is **very ample** if there exists a finite number of global sections $s_0, \ldots, s_n$ of $\mathcal{L}$ with no common zeroes such that the morphism

$$\begin{aligned}[s_0, \ldots, s_n]\colon X &\to \mathbb{P}^n \\ x &\mapsto [s_0(x) : \cdots : s_n(x)]\end{aligned}$$

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- $\mathcal{L}$ is **ample** if $\mathcal{L}^{\otimes N}$ is very ample for some $N > 0$.

Motivating Example

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In particular, notice: $H(1) = 1$ and $H(\frac{100001}{100000}) = 100001$.

Extending this to our setting:

Height for Rational Points in $\mathbb{P}^n$

- For $x = (x_0 : \cdots : x_n) \in \mathbb{P}^n(\mathbb{Q})$, define $H(x) = \max_{0 \le i \le n} |x_i|$.

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- For $x = (x_0 : \cdots : x_n) \in \mathbb{P}^n(\mathbb{Q})$, define $H(x) = \max_{0 \le i \le n} |x_i|$.
- For general number field k:

$$
\begin{aligned}
H\colon \mathbb{P}^n(k) &\longrightarrow \mathbb{R}_{>0} \\
x &\longmapsto \prod_v H_v(x).
\end{aligned}
$$

where $H_v(x) := \max_{0 \le i \le n} |x_i|_v$ for each place v of k.

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Northcott Property: If $\mathcal{L}$ is ample, then $\{x \in X(k) \mid H(x) \leq B\}$ is finite for any B.
Generalising Ampleness: Instead of *ampleness*, one may wish to consider *big and nef* line bundles, see [LT19].

Pseudo-Effective Cone

Let X be a smooth projective Fano variety.

- Define

$$N^1(X) := \mathrm{Pic}(X) \otimes \mathbb{R}$$

to be the **Néron-Severi space** of X.[a]

[a] Actually, it should be $N^1(X) := N^1(X)_{\mathbb{Z}} \otimes \mathbb{R}$ where $N^1(X)_{\mathbb{Z}}$ is the group of Cartier divisors modulo numerical equivalence, but this coincides with $\mathrm{Pic}(X)$ when X is smooth projective Fano.

[b] Recall: a **convex cone** C is a vector space such that $x, y \in C$ implies $\alpha x + \beta y \in X$ for any positive scalars α, β.

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to be the **Néron-Severi space** of X.[a]
- The **pseudo-effective cone of divisors**, denoted $\overline{\mathrm{Eff}}^1(X)$, is the closure in $N^1(X)$ of the convex cone[b] generated by classes of effective divisors.

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Let X be a smooth projective variety, $\mathcal{L}$ an ample divisor. The a-**invariant** is defined as

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Examples

- Let X be Fano and $\mathcal{L} = -K_X$. Then $a(X, \mathcal{L}) = 1$.
- Let $X = \mathbb{P}^n$, and $\mathcal{L} = H$. Since $K_X = -(n+1) \cdot H$, deduce $a(X, \mathcal{L}) = n + 1$.
- Let $X_d \subset \mathbb{P}^n$ be smooth hypersurface of degree d, and $\mathcal{L} = H$. Then $a(X, \mathcal{L}) = n + 1 - d$.

Geometric Invariants: b-invariant

b-invariant

Let X be a nice[a] variety over k, and $\mathcal{L}$ be an ample divisor on X. Suppose $a(X, \mathcal{L}) > 0$. Then, the b-**invariant** is defined[b] as

$$b(X, \mathcal{L}) = \begin{array}{c}\text{codim. of the minimal face}\\ \text{of } \overline{\mathrm{Eff}}^1(X) \text{ containing } K_X + a(X, \mathcal{L})\mathcal{L}\end{array}$$

[a] Smooth, projective, geometrically integral.

[b] Actually, it should be defined as the codimension of the minimal *supported* face, but this subtlety doesn't matter in our setting; see [LT19, §4].

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Ex: $\mathcal{L} = -K_X$

Let X be a smooth Fano variety, and $\mathcal{L} = -K_X$. Then $b(X, \mathcal{L}) = \dim \mathrm{Pic}(X)$.

Ex: Blowup of $\mathbb{P}^2$ at one point

Let X be the blow-up of $\mathbb{P}^2$ at a point, and $\mathcal{L}$ be an ample divisor. Denote H to be the pullback of hyperplane class on $\mathbb{P}^2$ and E the exceptional divisor.

- $a(X, \mathcal{L}) = \max\{\frac{3}{r+s}, \frac{2}{r}\}$.

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- $a(X, \mathcal{L}) = \max\{\frac{3}{r+s}, \frac{2}{r}\}$.
  [Why? **Step 1:** Check that

  $\overline{\mathrm{Eff}}^1(X)$ is spanned by E and $H - E$.

  $\mathrm{Nef}^1(X)$ is spanned by H and $H - E$.[a]

  Since $\mathcal{L}$ is ample, it is big and nef, and so $\mathcal{L} = rH + s(H - E) = (r + s)H - sE$ for some $r > 0$ and $s \geq 0$.

[a] $\mathrm{Nef}^1(X)$ is defined as the dual cone of pseudo-effective cone of under intersection pairing.

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  Step 2: We know $K_X = -3H + E$. Hence, for $K_x + t\mathcal{L} \in \overline{\mathrm{Eff}}^1(X)$, this imposes the conditions $t(r + s) - 3 \geq 0$ and $tr - 2 \geq 0$. Hence, deduce that $t = \max\{\frac{3}{r+s}, \frac{2}{r}\}$.]

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- $b(X, \mathcal{L}) = \begin{cases} 2 & \text{if } \mathcal{L} \text{ is proportional to } -K_X, \text{ i.e. } r = 2s \\ 1 & \text{if otherwise.} \end{cases}$.

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[Why? **Suppose** $r = 2s$. Then $t = \frac{3}{r+s} = \frac{2}{r} = \frac{1}{s}$, and so

$$K_x + t\mathcal{L} = (t(r+s) - 3)H + (1 - ts)E = 0.$$

Since $\overline{\mathrm{Eff}}^1(X) = \langle E, H - E \rangle$, deduce $b(X,\mathcal{L}) = 2$.
Suppose instead $r \neq 2s$. Then either $t = \frac{3}{r+s}$ or $t = \frac{2}{r}$, and so either $K_x + t\mathcal{L} = (\frac{3s}{r+s} - 1)E$ or $K_x + t\mathcal{L} = (-1 + \frac{2s}{r})(H - E)$. In either case, deduce $b(X,\mathcal{L}) = 1$.]

Manin's Conjecture

Setup:

- X is a suitable[a] Fano variety over a number field k.
- $\mathcal{L}$ is an ample line bundle on X, and $H_{\mathcal{L}}\colon X(k) \to \mathbb{R}_{>0}$ is the height function associated to $\mathcal{L}$.
- For any subset $Q \subset X(k)$, define the counting function

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Then, there exists an exceptional set $Z \subset X(k)$ such that

$$N_{\mathcal{L}}(X(k) \setminus Z, B) \sim cB^a \log B^{b-1}, \qquad B \to \infty,$$

where c is Peyre's constant.

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Example ([Bro09, Thm 1.2]): Projective Spaces over $\mathbb{Q}$

Let $n \geq 2$ and let $H \colon \mathbb{P}^{n-1}(\mathbb{Q}) \to \mathbb{R}_{>0}$ metrised by $\mathbf{z} = \max_i |z_i|$. Then,

$$N(\mathbb{P}^{n-1}, B) = \underbrace{\frac{2^{n-1}}{\zeta(n)}}_{\text{constant}} \cdot B^n + \underbrace{O_n(B^{n-1}(\log B)^{b_n})}_{\text{error term}}$$

Example ([Ser97, §2.12]): Blowup of $\mathbb{P}^2$ at a Point

Let X denote the blowup of $\mathbb{P}^2$ at a point, $\mathcal{L}$ be the ample line bundle/divisor, E be the exceptional divisor. If $U := X \setminus E$, then

$$N_{\mathcal{L}}(U, B) = \begin{cases} cB^a & \text{if } \mathcal{L} \text{ is not proportional to } -K_X \\ cB^a \log B & \text{if } \mathcal{L} \text{ is proportional to } -K_X \end{cases}$$

What's known?

Here's an incomplete list of varieties for which Manin's Conjecture has been verified for:

- Toric varieties [BT98]
- Smooth complete intersections of small degree in $\mathbb{P}^n$ [Bir62]
- Generalised flag varieties [FMT89]
- Nice compactifications of suitable groups (e.g. equiv. compactifications of $\mathbb{G}_a^n$ [CLT02], wonderful compactifications of semi-simple groups of adjoint type [GMO08] etc.)

What's in our toolbox?

- The Circle Method
- Harmonic Analysis
- Universal Torsor Method
- Dynamics & Ergodic Theory

For a more detailed overview, see e.g. [TB09, §3.5].

What's in our toolbox?

- **The Circle Method**
- Harmonic Analysis
- Universal Torsor Method
- Dynamics & Ergodic Theory

For a more detailed overview, see e.g. [TB09, §3.5].

Manin's Conjecture: Structure vs. Randomness

Manin's Conjecture splits into two sub-problems:

Sub-problem #1: Identify the exceptional set

Sub-problem #2: Bound the growth of the remaining points

Defining the Local Region of Chaos

We would like to throw out a small subset of $X(k)$ that throws off the asymptotic formula: but what does it mean to be a "*small*" subset of $X(k)$?

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Originally, it was conjectured that the exceptional set was some non-Zariski dense subset, but a counter-example was discovered by Batryev-Tschinkel [BT96].

Defining the Local Region of Chaos

We would like to throw out a small subset of $X(k)$ that throws off the asymptotic formula: but what does it mean to be a "*small*" subset of $X(k)$?

Peyre [Pey03] later suggested that the exceptional set should be a *thin set*:

Thin Set

A **thin set** of $X(k)$ is a finite union of subsets $\cup_i Z_i \subset X(k)$, where each Z_i comes from one of the following two sources:

Type I: Bad subvariety: $Z_i \subset Y_i(k)$ for a proper subvariety $Y_i \subsetneq X$

Type II: Bad covering: $Z_i = f(Y_i(k))$ for $f \colon Y_i \to X$ a generically finite map of degree ≥ 2.

For more details on thinness, see [Ser97, §9.1]. Lehmann-Sengupta-Tanimoto [LST22] have done some work analysing the geometry of these exceptional thin sets; Peyre [Pey21] has asked if "freeness" of rational points could replace the thinness condition, but there are subtleties [Saw].

In [Pey95], Peyre proposed the following refinement of Manin's Conjecture.

Manin-Peyre Equidistribution Principle

Set-up:

- X is an almost-Fano variety over number field k.
- ω^X is the Tamagawa measure on $X(\mathbb{A}_k)$ induced by the fixed adelic metric on $-K_X$.
- For every infinite subset $M \subset X(k)$, we define the counting measure on $X(\mathbb{A}_k)$:

$$\delta_{M \leq B} = \sum_{P \in M : H(P) \leq B} \delta_P$$

Then, there exists a thin subset $Z \subset X(k)$ such that we have a vague convergence of measures on $X(\mathbb{A}_k)$ as $B \to \infty$:

$$\frac{1}{B(\log B)^{r-1}} \cdot \delta_{(X(k) \setminus Z)_{\leq B}} \rightsquigarrow \alpha(X) \cdot \beta(X) \cdot \omega^X$$

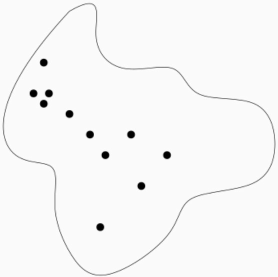
Manin's Conjecture & Equidistribution (Peyre)

????

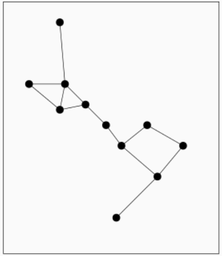
More on this in Week 2!

Pseudo-Randomness in Graph Theory

Motivation



Rational Points on a Variety



Graph

In the Intro, I said that once we start viewing rational points as vertices of our graph, we can start asking questions about the presence/absence of configurations in the graph, or about the pseudo-random behaviour of the edges. This section will develop this remark, and explain why this is interesting.

Question: What does it mean for a graph to have its edges distributed in a random-like manner?

Erdös-Rényi Graph $G(n, p)$

The **Erdös-Rényi graph** $G(n, p)$ considers a set of n vertices and decides with independent probability p whether any pair of vertices $\{v, v'\}$ has an edge.

Randomness & Random-like

Observation: Strictly speaking, the Erdös-Rényi model isn't a single graph but a machine that spits out graphs with n vertices. Nonetheless, we can still ask how the *generic random graph* looks like.

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Notice: the Erdös-Rényi model induces a **probability distribution** on the space of graphs with n vertices. As such, we can define:

Random-like Property

Given some graph property $\mathcal{P}$, let G_n be a graph with n vertices. We say $G(n, p)$ **almost always satisfies** $\mathcal{P}$ if

$$\Pr(G_n \mid G_n \text{ satisfies } \mathcal{P}) \to 1 \qquad \text{as } n \to \infty$$

The following theorem is a landmark result by Chung-Graham-Wilson [CGW89] proving the equivalence of various random-like properties ("quasi-randomness"):

Quasi-Random Graphs

Fix $p \in [0,1]$. Define G_n be a graph with n vertices and $(p + o(1))\binom{n}{2}$ edges; denote $G := \{G_n\}$ to be a sequence of such graphs.

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Quasi-Random Graphs

Fix $p \in [0,1]$. Define G_n be a graph with n vertices and $(p + o(1))\binom{n}{2}$ edges; denote $G := \{G_n\}$ to be a sequence of such graphs. Then, TFAE:

1) *Discrepancy:* $e(X, Y) = p|X||Y| + o(n^2)$ for all $X, Y \subseteq V(G)$.
2) *Discrepancy':* $e(X) = p\binom{|X|}{2} + o(n^2)$ for all $X \subseteq V(G)$.
3) *4-cycles:* The number of labelled 4-cycles is at most $(p^4 + o(1))n^4$.
4) *Codegree:* Denote $\mathrm{codeg}(u, v)$ to be the # of common neighbours of u and v. Then

$$\sum_{u,v \in V(G)} |\mathrm{codeg}(u, v) - p^2 n| = o(n^3).$$

$\vdots$

Not all graphs are quasi-random, but a key result ("Szemeredi Regularity") essentially says that any dense finite graph is locally quasi-random.

Edge Density

Let $X, Y \subset v(G)$ be subsets of vertices of graph G. Denote $e(X, Y)$ to be the number of edges between X and Y:

$$e(X, Y) := |\{(x, y) \in X \times Y \mid xy \in E(G)\}|.$$

Define the **edge density** between X and Y by

$$d(X, Y) := \frac{e(X, Y)}{|X||Y|}.$$

Not all graphs are quasi-random, but a key result ("Szemeredi Regularity") essentially says that any dense finite graph is locally quasi-random.

Epsilon Regular Pair

Let G be a graph, and $U, W \subseteq v(G)$. Call (U, W) an ϵ-**regular pair** in G if for all $A \subseteq U$ and $B \subseteq W$ with $|A| \geq \epsilon|U|$ and $|B| \geq \epsilon|W|$, one has

$$|d(A, B) - d(U, W)| \leq \epsilon$$

Epsilon Regularity

Szemerédi's Regularity Lemma (1978)

For every $\epsilon > 0$, there exists $N(\epsilon)$ s.t. every finite graph G may be partitioned into m classes $V_1 \cup \cdots \cup V_m$ where $m \leq N$ and

- All of the pairs V_i, V_j satisfy $||V_i| - |V_j|| \leq 1$.
- All but at most ϵm^2 of the pairs (V_i, V_j) are ϵ-regular.

Two Sources of Chaos

Question: Where might we find irregularities in edge distribution within the Szemerédi partition of the graph?

I. Within each component of the Szemerédi Partition.

II. The possible existence of (a small percentage of) ϵ-irregular pairs.

> *... [Szemerédi] raises the question if the assertion of the lemma holds when we do not allow any irregular pairs in the definition of a regular partition. This, however is not true, as observed by several researchers, including L. Lovász, P. Seymour, T. Trotter and ourselves. A simple example showing irregular pairs are necessary is [the half graph].*
>
> — **Alon et. al. [ADL$^+$94]**

Half-Graph

A **half-graph** is a bipartite graph with vertex classes $A = \{a_1, \dots a_n\}$ and $B = \{b_1, \dots b_n\}$ in which $a_i b_j$ is an edge iff $i \leq j$.

Half-Graphs & Stable Regularity

Here's another surprising result. Using ideas from model theory, Malliaris-Shelah [MS14] prove that existence of irregular pairs is *entirely* due to the presence of half-graphs.

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For $r \in \mathbb{N}$, call a graph G r-**edge stable** if it contains no half-graph of length r. Then:

Stable Regularity

For each $\epsilon > 0$ and $r \in \mathbb{N}$, there exists $N = N(\epsilon, r)$ s.t. for any sufficiently large finite r-edge stable graph G, for some ℓ with $\ell \leq N$, the graph G can be partitioned into dijsoint pieces $A_1, \dots, A_\ell$ and:

- The partition is equitable, i.e. the sizes of the pieces differ by at most 1.
- **All** pairs are ϵ-regular, and moreover have density either $> 1 - \epsilon$ or $< \epsilon$.
- $N < (\frac{4}{\epsilon})^{2^{r+3}-7}$.

Bird's Eye View: Structure vs. Randomness

Recall Terence Tao's observation regarding how we often seek to understanding objects by breaking them into a "structured" component and a "pseudo-random" component, and that many (often deep) conjectures take the following form:

Conjecture

"Natural" objects behave pseudo-randomly after accounting for all obvious structures.

Bird's Eye View: Structure vs. Randomness

	Manin's Conjecture	Szemerédi Regularity
Object	Set of rational points $X(k)$	Graph G
Good/Uniform Behaviour	Equidistribution of Rational Points outside Exceptional Set	ϵ-regular pairs
Chaos/Irregular Behaviour	Accumulation of Rational Points within Exceptional Set	ϵ-irregular pairs
Chaos due to Structure	Thin Set	Half-Graphs
Smallness of Structured component	Thinness (generalising non-Zariski density)	At most ϵm^2 many ϵ-irregular pairs

Investigation Begins

Study Group Syllabus

- **Week 1:** Overview (Ming Ng)
- **Week 2:** Equidistribution & Peyre's Constant (Sebastian Monnet)
- **Week 3:** Manin's Conjecture for del Pezzo Surfaces
- **Week 4:** The Circle Method & Cubic Surfaces (Simon Myerson)
- **Week 5:** Repulsion of Rational Points
- **Week 6:** Szemerédi Regularity & Roth's Theorem
- **Week 7:** Roth's Theorem via the Circle Method
- **Week 8:** Graphons & Regularity

On Equidistribution. Manin's Conjecture & Szemerédi Regularity indicate two different ways of articulating uniform distribution of rational points:

- Rational points are *equidistributed* in the sense of Peyre [Pey95].
- Rational points are uniformly spread out in the sense that the edges of the graph (indicating closeness) are ϵ-*regular*.

Deeper Dive into Manin's Conjecture (Weeks 2-3)

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How do these two different flavours of equidistribution interrelate? To what extent are they aligned or misaligned?

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Natural Question

How do these two different flavours of equidistribution interrelate? To what extent are they aligned or misaligned?

…exploration begins in **Week 2**, where we delve into Peyre's Equidistribution Property & its relation to Manin's Conjecture.

Deeper Dive into Manin's Conjecture (Weeks 2-3)

On Del Pezzo Surfaces. We have a complete classification of Fano Varieties of dimension 2 ("del Pezzo surfaces"), yet Manin's Conjecture is not settled even in this setting. Understanding the current state of our knowledge will be the goal of **Week 3**.

Erdös-Turán Conjecture (1936)

Every subset $A \subseteq \mathbb{N}$ of the natural numbers with *positive upper density* contains k-term *arithmetic progressions* for every positive integer k.

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Every subset $A \subseteq \mathbb{N}$ of the natural numbers with *positive upper density* contains k-term *arithmetic progressions* for every positive integer k.

- A subset $A \subseteq \mathbb{N}$ is said to have **positive upper density** if

$$\limsup_{n\to\infty} \frac{A \cap \{1, 2, \dots, n\}}{n} > 0.$$

- An **arithmetic progression** is a sequence $\{a_n\}_{n\in\mathbb{N}}$ of the form

$$a_n = a_1 + (n-1)d.$$

e.g. a 3-term arithmetic progression is of the form $a, a+d, a+2d$.

Erdös-Turán Conjecture (1936)

Every subset $A \subseteq \mathbb{N}$ of the natural numbers with *positive upper density* contains k-term *arithmetic progressions* for every positive integer k.

Roth's Theorem (1953)

Settled the case for $k = 3$ via an adaptation of the Circle Method, but found difficulties extending his approach to arbitrary k.

Szemerédi's Theorem (1975)

Settled the case for $k = 4$ in 1969, and then for arbitrary k in 1975, leveraging the newly developed tool of Szemerédi Regularity.

Perspectives on the Circle Method (Weeks 4, 6 & 7)

The Circle Method has played an important role in illuminating our understanding of rational points on cubic forms – given our previous discussion, this is a natural place to start looking for clues on how to apply our graph-theoretic idea.

Week 4: Circle Method & Cubic Forms. This week will introduce the Circle Method and its applications to cubic surfaces. Time permitting, we may discuss conditional vs. unconditional results on cubic forms using the Circle Method.

Week 6: Szemerédi Regularity & Roth's Theorem. Prove Roth's Theorem via Szemerédi Regularity – for a reference, there's an online MIT course uploaded on YouTube with accompanying lecture notes [Zha19a].

Week 7: Roth's Theorem via the Circle Method. Prove Roth's Theorem via Roth's original approach. Again, the same MIT course has online lecture videos on this and accompanying lecture notes [Zha19c].

Repulsion Principles

In order to draw our graph of rational points, we need to figure out when to draw an edge or not between the vertices.

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Repulsion on curves C (Mumford [Mum65])

Let C be a smooth proj. curve over number field K where $g(C) \geq 2$. Order the finite set of k-rational points of C by increasing height $x_1, x_2, \dots$. Then,

$$H(x_i) \geq e^{ai+b}, \qquad a, b \in \mathbb{R}_{>0}.$$

Repulsion Principles

In order to draw our graph of rational points, we need to figure out when to draw an edge or not between the vertices. So how to determine when two rational points are "close"? Thinking about how rational points tend to repel each other seems relevant.

Repulsion on varieties with Kodaira dim ≥ 0 (McKinnon [McK11])

Set-up:

- Let X be a smooth proj. variety over number field k with non-negative Kodaira dimension.
- Assume that appropriate subvarieties of $X \times X$ behave nicely (e.g. those of non-negative Kodaira dimension satisfy Vojta's Conjecture).

Then, for any $\epsilon > 0$, there exists a non-empty Zariski open subset $U(\epsilon)$ of X and a positive real constant C s.t.

$$\mathrm{dist}_v(P,Q) > CH(P,Q)^{-\epsilon}, \qquad \text{for all } P,Q \in U(k).$$

Repulsion Principles

Week 5: Repulsion of Rational Points. We will discuss McKinnon's result [McK11], which proves a repulsion principle (conditional on Vojta's Conjecture). This was later used to prove Batryev-Manin's Conjecture for $K3$ surfaces.

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Remark: Mumford's result is for curves of general type (i.e. $g(C) \geq 2$), McKinnon's result is for varieties of non-negative Kodaira dimension (e.g. $K3$ surfaces) – not Fano varieties. The question of how to extend these repulsion principles to the Fano setting seems to be a subtle one, but see [Fat] and [Hua17].

A couple of orienting remarks:

- McKinnon's result says that two rational points that are close to each other (e.g. w.r.t. some Archimedean metric), then they must have large height relative to this distance.
- Szemerédi Regularity, at least classically, is applied to dense *finite* graphs; however, we expect $X(k)$ to be *infinite* for X Fano.

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It may therefore be more helpful to think of $X(k)$ not as a single static graph, but as a dynamic sequence of finite graphs.

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Characteristic Sequence

Let X be a suitable Fano variety over k, with ample line bundle $\mathcal{L}$ and associated height function $H_{\mathcal{L}}$. Define the **characteristic sequence** of X to be the family of finite[a] graphs $\{G_n\}_{n\in\mathbb{N}}$ whereby:

- *Vertices of G_n:* k-rational points of X of bounded height $\leq n$.
- *Edges of G_n:* Draw an edge everytime two k-rational points are close.

[a] Notice: G_n is a finite graph for all n due to the Northcott Property.

Some natural questions: How do we analyse sequences of graphs? What might we say about their behaviour in the limit as $n \to \infty$? Where does Szemerédi Regularity figure in this new setting?

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Week 9: Graphons & Regularity Introduce the notion of graphons as the limit of a sequence of graphs, and explain connections with Szemerédi Regularity. Again, we will use the MIT online course as a reference – see [Zha19b]

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